2FTC

If f is continuous,

$$F(x) = \int_a^x f(t)\,dt, \text{ then}$$

must be a constant

$$F'(x) = f(x).$$

---

example:

Let $G(x) = \int_{x^2}^{3e^x} \cos(u^3)\,du$

Find $G'(x)$

Solution:

$$G(x) = \int_1^{3e^x} \cos(u^3)\,du + \int_{x^2}^1 \cos(u^3)\,du$$

$$\left[\int_a^b + \int_b^c = \int_a^c \right]$$

$$= \int_1^{3e^x} \cos(u^3)\,du - \int_1^{x^2} \cos(u^3)\,du$$

$$\left[\int_b^a = -\int_a^b \right]$$

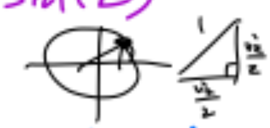
2FTC: $G'(x) = \cos\left(\left(3e^x\right)^3\right) \cdot \left(3e^x\right)'$

$$- \cos\left(\left(x^2\right)^3\right) \cdot \left(x^2\right)'$$

$$\Rightarrow \boxed{G'(x) = 3e^x \cos\left(27e^{3x}\right) - 2x\cos\left(x^6\right)}$$

Example: Simplify $\sin\left(\arccos(x) + \frac{\pi}{4}\right)$.

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$



$$\Rightarrow \sin\left(\arccos(x) + \frac{\pi}{4}\right) = \sin(\arccos(x))\underbrace{\cos\left(\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}} + \underbrace{\cos(\arccos(x))}_{x}\underbrace{\sin\left(\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}}$$

$\theta = \arccos(x)$

$\cos\theta = x = \frac{x}{1}$

$\sin\theta = \sqrt{1-x^2}$

$$= \boxed{\left(\sqrt{1-x^2}\right)\frac{\sqrt{2}}{2} + x\,\frac{\sqrt{2}}{2}}$$

Rewrite using partial Fractions

$$\frac{2x^2+3x+6}{(x+1)(x+2)(x^2+4)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

Using this equation → solve for A, B, C, D.

To solve: multiply by common denom

$$2x^2+3x+6 = A(x+2)(x^2+4) + B(x+1)(x^2+4)$$
$$+ (Cx+D)(x+1)(x+2)$$

At this point: need equations for A, B, C, D

coefficient of x^3 term: $0 = A + B + C$ (1)

Next: • plug in different #s for x to get more equations.

or • Multiply RHS out completely, and The x^2, x, constant terms must match on both sides.

Constant term on both sides $(x=0)$

$$6 = A(2)(4) + B(1)(4) + D(1)(2)$$

(2) $6 = 8A + 4B + 2D.$

Plug $x=-2$ (so $(x+2)=0$)

$$\Rightarrow 2(-2)^2 + 3(-2) + 6 = 0 + B(-1)((-2)^2+4) + 0$$

$\Rightarrow \quad 8 = B(-8) \Rightarrow \boxed{B=-1}$ (3)

Plug $x=-1$ (so $(x+1)=0$)

$\Rightarrow 2(-1)^2+3(-1)+6 = A(1)((-1)^2+4)+0+0$

$\Rightarrow \quad 5 = A\cdot 5 \Rightarrow \boxed{A=1}$ (4)

(1) $A+B+C=0$

$\Rightarrow \quad 1+(-1)+C=0 \Rightarrow \boxed{C=0}$

(2) $6 = 8A+4B+2D.$

$6 = 8\cdot 1+4(-1)+2D$

$6 = 4+2D$

$2 = 2D \Rightarrow \boxed{D=1}$

$$\Rightarrow \frac{2x^2+3x+6}{(x+1)(x+2)(x^2+4)} = \frac{A}{x+1}+\frac{B}{x+2}+\frac{Cx+D}{x^2+4}$$

$$= \boxed{\frac{1}{x+1}+\frac{-1}{x+2}+\frac{1}{x^2+4}}$$

(can check using partial_fraction command in sagemath:

f(x).partial_fraction().

General format for partial fractions:

degree 3 $\quad 3<10$ ✓

$$\frac{x^3-3x+7}{(x+1)^3(x^2+x-1)(x^2+1)^2(x-3)}$$

degree 10

If top deg ≥ bottom deg then first do long division

$$= \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{Dx+E}{x^2+x-1}$$

$$+ \frac{Fx+G}{x^2+1} + \frac{Hx+I}{(x^2+1)^2} + \frac{J}{x-3}$$

Example Find $\displaystyle\int \frac{4x+8}{(x+1)(x+3)}\,dx$

deg 1

deg 2

partial fractions: $\displaystyle\frac{4x+8}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

$$\Rightarrow 4x+8 = A(x+3) + B(x+1)$$

x coefficients on both sides

$\Rightarrow 4 = A + B \ (1)$

Set $x = 0$

$\Rightarrow 8 = 3A + B \ (2)$

$$\begin{array}{rl} & 3A + B = 8 \\ \text{subtract} & A + B = 4 \\ \hline & 2A = 4 \Rightarrow A = 2 \end{array}$$

$2 + B = 4$

$B = 2$

$$\int \frac{4x+8}{(x+1)(x+3)}\,dx = \int \left(\frac{2}{x+1} + \frac{2}{x+3}\right) dx$$

$$= \boxed{2\ln|x+1| + 2\ln|x+3| + C}$$

$$\ln\left|(x+1)^2(x+3)^2\right| + C$$

$\boxed{\text{Ex}}$ $\displaystyle\int \frac{x^3+1}{x(x^2+1)}\,dx$

deg 3

deg 3

$x^3 + x$

long division first

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad\quad 1 \\ \hline x^3+0x^2+x+0 & x^3 + 0x^2 + 0x + 1 \\ & -(x^3 + 0x^2 + x + 0) \\ \hline & \quad\quad\quad\quad -x + 1 \end{array}$$

$$\Rightarrow \frac{x^3+1}{x(x^2+1)} = 1 + \frac{-x+1}{x(x^2+1)}$$ part. f.

$$\frac{-x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \quad -x+1 = A(x^2+1) + (Bx+C)x$$

x^2 term : $0 = A + B$

$x=0$ $\quad 1 = A \Rightarrow A=1$

$B=-1$

$x=1$ $\quad 0 = A\cdot 2 + (B+C)1$

$0 = 2 + (-1+C) = 1+C$

$C=-1$

$$\frac{-x+1}{x(x^2+1)} = \frac{1}{x} + \frac{-x-1}{x^2+1}$$

$$\int \frac{x+1}{x(x^2+1)}dx = \int 1 + \frac{1}{x} + \frac{-x}{x^2+1} - \frac{1}{x^2+1}\,dx$$

$u=x^2+1$

$du = 2x\,dx$

$$= \boxed{x + \ln|x| - \frac{1}{2}\ln|x^2+1| - \arctan(x) + C}$$